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A MATHEMATICAL MODEL OF THE DEFORMATION AND FAILURE OF ROCK MATERIALS*

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A mathematical model is proposed for the deformation and failure of rock materials in which the bulk plastic deformation rate includes hydrostatic and dilatancy components. The latter is taken to be proportional to the rates of change of the stress tensor invariants. Shear is described by plastic flow theory relationships analogous to /1-3/. The model relationships do not take account of effects associated with the influence of the strain rate on the fracture process. Therefore, the model proposed can be considered as the limit, dynamic or static. It is conceivable that the quantitative measure of the effects taken into account and the specific form of the main dependences for the two limit models will be distinct for an identical material.

Within the framework of the model proposed, quantitative data are presented on the mechanical characteristics of different mountain rocks obtained on the basis of the authors' statistical treatment of static test results published in the literature.

A number of mathematical mountain rock models have been proposed /1, 4-9/ that reflect available experimental data on their deformation and fracture under static and dynamic loads to some degree /10-22/. Within the framework of these models the mountain rocks are considered as a continuous medium in the elastic and plastic stages of operation as well as in the fractured state. It is assumed here that considerable changes occur in the state of stress of such a medium at distances considerably exceeding the block dimensions into which the medium is separated in the developed fracture stage.

As is seen from the experimental data /10-22/, the important effects observed during fracture are the change in strength and the occurrence of additional porosity in connection with the appearance of block disintegration, the appearance of so-called dilatancy of the medium.

It is assumed in the formulation of the mathematical models for such media in /6-9/ that the rate of the dilatancy component of the bulk strain is proportional to the shear rate. The rate of the dilatancy component of the bulk strain in /7, 9/ depends additionally on the first invariant of the stress tensor.

1. Let us consider the available experimental data characterizing the fracture process for specimens of sufficiently strong rocks.

Fig.1 shows results of testing grey sand specimens with density $\gamma = 2.49$ g/cm^3, porosity $w = 5.9\%$ under non-proportional loading conditions /15/ ($\sigma_1, \sigma_2 = \sigma_3 = \text{const}, \sigma_1, \sigma_2, \sigma_3$ are the principal stresses). Here $\sigma = {}^1/_3\,(\sigma_1 + 2\,\sigma_2)$, ε is the bulk strain. Tests were performed on a "rigid" apparatus which enabled us to obtain the descending section on the material strain diagram. The values of σ_2 varied during testing between 3 and 150 MPa in different tests.

The presence of three characteristic points corresponding to the appearance of plastic deformations, the achievement of the maximum specimen strength, i.e., the maximum stress σ_1^{max}, and the achievement of the residual strength when the increase in the bulk strain ceases, should be noted.

An analogous phenomenon is also observed for conditions of proportional specimen loading ($\sigma_2/\sigma_1 = \text{const}, \sigma_2 = \sigma_3$) /10/.

The geometric loci of these three classes of points in the $\sqrt{J_2}, \sigma$ plane will be curves corresponding to the conditions of the initial, maximum, and residual strength, which are obviously independent of the loading trajectory /17, 18/.

Processing the results of experiments /12-14/, conducted with hollow and solid cylindrical specimens of different mountain rocks under conditions of simultaneous compression

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and torsion $(\sigma_1 \neq \sigma_2 \neq \sigma_3)$ showed, in particular, that the curves of the initial and maximum strength can also be considered to be independent of the third stress tensor invariant I_3 $\sigma_1 \sigma_2 \sigma_3$. The corresponding maximum strength curves for certain rocks are presented, according to the data in /12/, in Fig.2 (curve *1* is for limestone, $\gamma = 2.60$ g/cm^3, $w = 4.7\%$, and curve *2* is for dolomite, $\gamma = 2.84$ g/cm^3, $w = 0.9\%$). The numbers *1-4* denote points obtained in tests for different values of $(I_3)^{1/3}$ (MPa) *1* — 0; *2* — 100 — 230; *3* — −50 — −130; *4* — −150 — −290.

The experiments show that a further growth of the "plastic" deformations of disintegration under decreasing stresses occurs after the maximum strength is reached. This process is terminated by reaching the residual strength curve in the $(\sqrt{J_2}, \sigma)$ plane which characterizes the ability of the fractured speciment still to carry a certain load /15, 16, 19/.

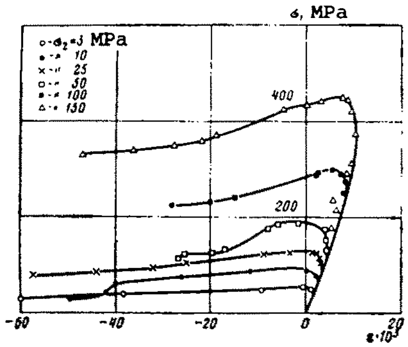

Fig.1

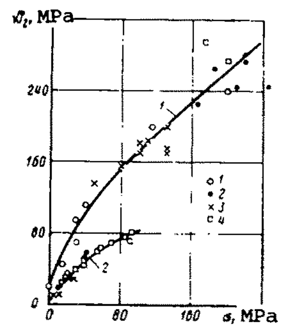

Fig.2

2. The experimental data presented above enable us to introduce three limit characteristics of the rocks, the initial strength condition $F_1(\sigma_{ij}) = 0$, the maximum strength condition $F_2(\sigma_{ij}) = 0$, and the residual strength condition $F_3(\sigma_{ij}) = 0$.

Assuming these conditions to be independent of the loading history and of the third stress tensor invariant as an essential hypothesis, we write them in the invariant form

$$\sqrt{J_2} = F_k(\sigma), \quad k = 1, 2, 3 \tag{2.1}$$
$$\sigma = I_1/3, \; I_1 = \sigma_{kk}, \; J_2 = I_2 - I_1^2/6 = {}^1/_2\, S_{ij}S_{ij}$$

where I_1, I_2 are the stress tensor invariants $S_{ij} = \sigma_{ij} - \sigma\delta_{ij}$ are the stress tensor deviator components, and σ_{ij} are the stress tensor components, $i, j = 1, 2, 3$.

We will introduce a relationship expressing the condition of plasticity in shear strain, which as is seen from the experimental results presented above will depend on the hardening characteristics in the general case. We will take the bulk plastic dilatancy deformation ε_D^p as the hardening parameter and we will use for the plasticity condition in shear

$$\sqrt{J_2} = F(\sigma, \varepsilon_D^p) \tag{2.2}$$

Taking account of the plasticity condition (2.2), we write the flow relationships for the plastic shear strain increments, as in /2/, in the form

$$Ge_{ij}' \equiv G\left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right) = \frac{dS_{ij}}{dt} + \lambda S_{ij} \tag{2.3}$$

$$\lambda = \frac{GW' - F\,dF/dt}{F^2}\, e(W')\, e(\sqrt{J_2} - F) \tag{2.4}$$

$$W' \equiv S_{ij}e_{ij}' \tag{2.5}$$

Here e_{ij} is the strain rate tensor; the remaining notation is standard /2/.

The definition of the shear strain unloading is also contained in these relationships: $e_{ij}'^p = 0$ should be in such an unloading, but by definition $e_{ij}'^p = \lambda S_{ij}/G$ in (2.3), therefore, $e_{ij}'^p = 0$ for $\lambda = 0$, and according to (2.4) this holds for $W' = 0$ and $\sqrt{J_2} < F$ since $e(u) = 1$ for $u \geqslant 0$ and $e(u) = 0$ for $u < 0$.

Furthermore, we shall assume that the plasticity condition (2.2) is associated with the limit characteristics (2.1) introduced above by the relationships

$$F = F_A = (F_1 - F_2)\, f_1(\varepsilon_D^p) + F_2 \tag{2.6}$$
$$F = F_B = (F_2 - F_3)\, f_2(\varepsilon_D^p) + F_3$$

Here F_A corresponds to the pre-limiting loading and strain states, i.e., the states being realized on changing from the initial to the maximum strength, and F_B correspond to the post-limiting, states i.e., the states obtained on changing from the maximum to the residual strength. The functions f_1, f_2 determined experimentally characterize the dependence of the shear plasticity condition on the hardening parameter ε_D^p.

3. We will now construct relationships governing the volume deformability of a medium. The volume strain rate $d\varepsilon/dt \equiv e = e_{kk}$ is represented in the form of a sum of the elastic component e^e and the plastic component e^p, while e^p is represented in the form of the sum of a "hydrostatic" component e_H^p and a dilatancy component e_D^p.

As in /2/, we take a deformation-type dependence

$$\varepsilon^e = \varepsilon_H - \varepsilon_H^p = \varphi(\sigma, \varepsilon_H^p) \tag{3.1}$$

for the elastic part ε^e of the bulk strain hydrostatic component ε_H.

The family of lines in the σ, ε^e plane governed by relationship (3.1) for $\varepsilon_H^p = \text{const}$ is the elastic strain law for hydrostatic loading. The cumulative hydrostatic component of the irreversible bulk strain is determined by the "kinetic" equation

$$\frac{d\varepsilon_H^p}{dt} = \frac{df^{-1}(\sigma)}{d\sigma}\left(\frac{d\sigma}{\partial t}\right) e\left(\frac{d\sigma}{dt}\right) e\left(\varepsilon^e - \varepsilon^e_{\max}\right) \tag{3.2}$$

$$\varepsilon^e_{\max} = \psi(\varepsilon_H^p)$$

and the function ψ is obtained from (3.1) for $\varepsilon^e = \varepsilon^e_{\max}$, $\sigma = \sigma_*$, where

$$\sigma_* = f(\varepsilon_H^*) = f(\varepsilon_H^p + \varepsilon^e_{\max}) \tag{3.3}$$

The function f (f^{-1} is the function inverse to f) in (3.2) and (3.3) governs the loading branch of the hydrostatic compression diagram of the medium.

The definition of unloading from the plastic deformation state during hydrostatic loading and unloading processes is also contained in the relationships presented above.

To determine the dilatancy component of the plastic strain ε_D^p we take a hypothesis expressed by the relationship

$$\frac{d\varepsilon_D^p}{dt} = A_j \frac{dJ_j}{dt} e(W^{\cdot}) \tag{3.4}$$

where $W^{\cdot}$ is the energy dissipation rate by the shear strains determined by means of (2.4), J_j ($j = 1, 2, 3$) are the stress tensor invariants, and A_j are certain functions of the quantities $J_j, \varepsilon_D^p, \varepsilon_H^p$ and possibly other parameters.

Definitions of the loading and unloading concepts for dilatancy strain are contained in relationship (3.4): $d\varepsilon_D^p > 0$ for $W^{\cdot} \geqslant 0$ for loading, and $d\varepsilon_D^p = 0$ for $W^{\cdot} < 0$ for unloading.

In general the relationship (3.4) is not integrable with respect to the variable J_j, where A_j can depend on the loading "history" in a complex manner, i.e., on the parameters $\varepsilon_D^p, \varepsilon_H^p$ and others and can also be distinct in the pre- and post-limit states.

In the special cases of simple loading trajectories, for instance for $\sigma_2 = \sigma_3$, $\sigma_3/\sigma_1 = \text{const}$ or for $\sigma_2 = \sigma_3 = \text{const}$ and increasing σ_1, as hold in the experiments discussed above, the relationship (3.4) can be integrated and reduced to the form

$$\varepsilon_D^p = \Phi(\sigma, \sqrt{J_2}, I_3, a_i) \tag{3.5}$$

where a_i are parameters characterizing this class of loading trajectories. The representation (3.5) will certainly be distinct for the pre- and post-limit loading modes.

The relationships constructed above form a closed model of a medium experiencing irreversible shear, bulk hydrostatic and bulk dilatancy strain. According to the model relationships, the mechanical energy dissipation (the work of the stress on the irreversible strain) is positive.

4. Processing of the available experimental results /10-22/ enabled us to make this model specific as follows. The function Φ characterizing the pre-limit strain of the medium was obtained in the form

$$\Phi_1 = \alpha\left(\beta \frac{\Delta\sqrt{J_2}}{\sigma - R_p}\right)^{\varkappa} \tag{4.1}$$

$$\beta = 1 + I_3^{1/3}/J_2^{1/2}, \quad \Delta\sqrt{J_2} = \sqrt{J_2} - (\sqrt{J_2})_e$$

where R_p is the strength of rocks under tension (compressive stresses are considered to be positive, i.e., $R_p < 0$); $\alpha, \varkappa$ are experimental coefficients presented in the table of different rocks, and also in Fig.3 as a function of values of the elastic wave propagation velocity a_0 in these rocks; $(\sqrt{J_2})_e$ is the value of $\sqrt{J_2}$ satisfying condition (2.1) for $k = 1$.

Mountain Rock	R_c, MPa / a_0, m/sec	γ, g/cm^3 / w, %	$\alpha \times 10^3$ / $\varkappa$	$\alpha_* \times 10^3$ / $\varkappa_*$	a_1 / ν_1	a_2 / ν_2	a_3 / ν_3
Quartzite [22]	281 / 6400	—	0.5 / 1,87	—	1,16 / 0.96	5,86 / 0.79	—
Sandstone P-O [10]	235 / 6060	2.76 / 0.36	3.6 / 2.71	—	3.15 / 0.86	5.61 / 0.79	—
Diorite [10]	232 / 6000	— / 0.15	2.65 / 3,04	—	2.65 / 0.88	6.21 / 0.79	—
Quartzite [19]	211 / 5850	—	10.4* / 4.3	17.5 / 0.62	1.45 / 0.99	1.65 / 0.98	2.02 / 0.93
Granodiorite [17]	207 / 5800	2.67 / 0.7	2.0 / 4.0	—	1.37 / 0.89	6.26 / 0.76	—
Granite [21]	263 / 4900	2.65 / 1.3	2.86 / 3.82	—	0.89 / 0.99	3.16 / 0.88	—
Diabase [10]	200 / 5750	2.97 / 0.98	5.31 / 2.0	—	3.01 / 0.88	3.92 / 0.85	—
Siltstone [10]	181 / 5600	—	3.18 / 2.09	—	5.0 / 0.78	7.7 / 0.75	—
Granite [15]	160 / 5300	—	40.6* / 3.11	31.12 / 0.57	2.23 / 0.9	1.86 / 0.94	1.04 / 1.0
Sandstone [15]	140 / 5100	2.49 / 5.9	76.4* / 2.81	16.15 / 0.52	6.6 / 0.75	6.0 / 0.78	3.84 / 0.82
Sandstone D-8 [10]	134 / 5030	2.49 / 7.4	1.89 / 4.92	—	2.44 / 0.88	8.88 / 0.72	—
Sandstone [17]	132 / 4900	2.45 / 8.5	2.5 / 5.0	—	2.5 / 0.83	8.79 / 0.71	—
Limestone [10]	79 / 4220	2.97 / —	5.6 / 4.61	—	2.24 / 0.88	11.31 / 0.65	—

The values of a_0 indicated in the table (except those cited according to /17, 21/) were determined from data in /23/ as a function of strength under uniaxial compression.

Figs.4a and b show as an example the results of determining the functions (4.1) from experimental data /10/ corresponding to the results of testing grey sandstone specimens with $\gamma = 2.76$ g/cm^3 and $w = 0.36\%$ under proportional loading conditions. The numbers *1-4* on the graphs denote the test data for the respective values of σ_2/σ_1 : *1* — 0; *2* — 0.069; *3* — 0.116; *4* — 0.178. It is seen that the experimental points corresponding to different values of the ratio σ_2/σ_1 are described well enough by one curve.

The post-limit plastic bulk strain is represented by the relationship

$$\Phi_2 = \varepsilon^p_{D*2} + \alpha_* \left(\frac{\delta\sqrt{J_2}}{\sigma - R}\right)^{\varkappa_*}, \quad R = R_p \frac{F - F_3}{F_2 - F_3} \qquad (4.2)$$

$$\delta\sqrt{J_2} = (\sqrt{J_2})_* - \sqrt{J_2}$$

where α_*, $\varkappa_*$ are the experimental coefficients (see the table), $(\sqrt{J_2})_*$ is the value of $\sqrt{J_2}$ when condition (2.1) is satisfied for $k = 2$, R and R is the strength of the mountain rock under tension, taking its variation during the fracture process into account.

Exactly as in the case of the pre-limit strain, the experimental points for different values of σ_2 (*1* — 3; *2* — 10; *3* — 25; *4* — 50; *5* — 100; *6* — 150 MPa) are described well by a single curve (Fig.5).

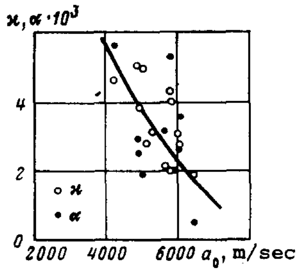

Fig.3

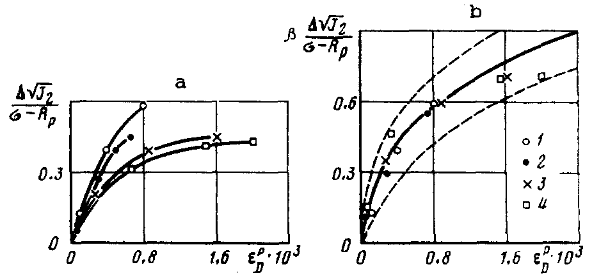

Fig.4

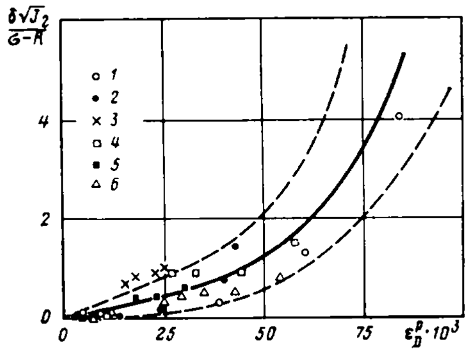

Fig.5

The functions $F_k(\sigma)$ are obtained in the form

$$F_k(\sigma) = a_k(\sigma - R_p)^{\nu_k}, \quad k = 1, 2, 3 \qquad (4.3)$$

where the values of a_k, ν_k are presented in the table for different rocks. The points *1* In Fig.6a and b are referred to the quantities ν_1, a_1 and the points *2* to ν_2, a_2.

The functions $f_1(\varepsilon_D{}^p)$ and $f_2(\varepsilon_D{}^p)$ are represented by the formulas

$$f_1 = 1 - \left(\frac{\varepsilon_D{}^p}{\varepsilon^p_{D*2}}\right)^{q_1}, \quad f_2 = 1 - \left(\frac{\varepsilon_D{}^p}{\varepsilon^p_{D*3}}\right)^{q_2} \qquad (4.4)$$

where ε^p_{D*2}, ε^p_{D*3} are values of the volumetric strain of disintegration corresponding to conditions (2.1) for $k = 2, 3$; $q_1 = 0.2$—0.5, $q_2 = 0.9$—1.2 are experimental coefficients for mountain rocks (see the table).

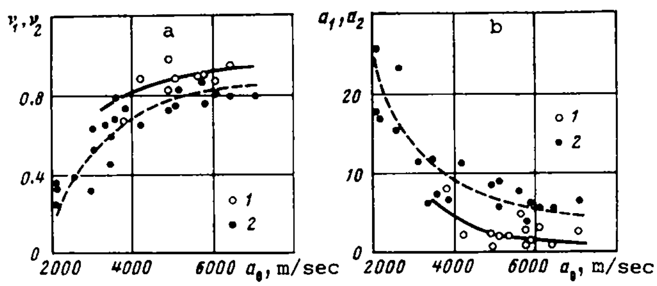

Fig.6

The results presented above for the processing (with the exception of those marked by asterisks in the table) indicate the sufficiently representative correlation dependence of the parameters α, $\varkappa$, a_1, a_2, a_3, ν_1, ν_2, ν_3 for the majority of the mountain rocks considered on the velocity of elastic wave propagation a_0 in these rocks.

The correlation dependences obtained can be utilized for preliminary estimates of the main mountain rock characteristics within the framework of the proposed model when data is available solely on the longitudinal elastic wave propagation velocity a_0.

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STRENGTH CRITERIA OF AN ANISOTROPIC MATERIAL*

B.E. POBEDRYA

Strength criteria are proposed for anisotropic materials as a generalization of the well-known phenomenological criteria for an isotropic medium based on the introduction of certain functions of the stress tensor invariants.

1. The viewpoint, according to which a composite is treated as a certain reduced homogeneous body /1, 2/, is well-known. If even each component of the composite is isotropic here, the reduced body possesses an anisotropy which is customarily called structural /2/.

A fairly large number of strength criteria, that agree to some extend with experimental data /3, 4/, have been developed for isotropic materials. The majority are based on the introduction of a certain function, which depends on the stress tensor, that describes a surface encompassing the safe stress states in the stress space

$$F(Y_1, Y_2, Y_3) = 0 \tag{1.1}$$

The function (1.1) should understandably depend on the temperature and possibly other parameters of a physicochemical nature. However, for simplicity we shall consider all these parameters to be fixed. Here Y_α $(\alpha = 1, 2, 3)$ are three independent invariants of a symmetric stress tensor /5/, for which we can select, say

$$Y_1 = \Theta = \sigma_{ii}, \; Y_2 = \sigma_u = (s_{ij}s_{ij})^{1/2}, \; Y_3 = \det|s_{ij}| \tag{1.2}$$

where σ_u is the intensity of the stress tensor $\|\sigma_{ij}\|$; summation from 1-3 is over repeated subscripts.

It is sometimes assumed that the function F is independent of the third invariant Y_3 and the criterion (1.1) is represented in the form

$$f(\sigma_u) = K(\Theta) \tag{1.3}$$

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